CAVITATION STREAMLINE FLOW OF LAMINA IN TRANSVERSE GRAVITATIONAL FIELD

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UDC 532.6+532.528

The problem of cavitation streamline flow located on the linear base of a lamina in a gravity solution current is solved by the systems of Ryabushinskii and Zhukovskii–Roshko. The method of fragment-continuum approximation of the boundary condition at the free boundary was used, in which this condition is exactly satisfied at a finite number of points. In this way the original problem comes down to a solution of a system of nonlinear equations whose solvability can be shown by the method of V. N. Monakhov [1]. The main consideration in the present work was given to a numerical solution of this system of equations on a computer. The problem is similar to the type for large Froude numbers, when the effect of weight on the flow is small, studied in [2-5]. In [6, 7] the flow problems were solved by the method of finite differences. The approximations of the boundary condition at the free boundary used earlier are based on the use of the smallness of these or other characteristics of flow. Thus, for example, the linearization of Levi–Chivit [8] is rightly used in the assumption of smallness of the change in the modulus and angle of inclination of the velocity at the free flow line; a stronger linearization is based on the requirement of smallness of additional velocities caused by an obstacle in comparison with the velocity of the undisturbed current [9]. In the given work the problems studied lead to a range of cavitation and Froude numbers when the gravitational force exerts a considerable effect on the main characteristics of the flow. As an example of one of the possible applications of the calculation, the solution of the problem of choice of the form of a body of zero buoyancy with a zone of constant pressure is given.

Diagrams of the currents under examination are presented in Fig. 1. For the characteristic sizes of the given sections the flow regions have a length l of the lamina DA and an angle $\alpha\pi$ which it forms with the infinite horizontal base CD. In Ryabushinskii's system the cavity is assumed to be symmetrical with respect to some vertical axis BC, which allows one to confine the study to one half of the flow region.

We select a conformal flow region in the physical plane $z = x + iy$ on the interior of the unit semicircle

$$|\zeta| < 1, \qquad \operatorname{Im}\zeta > 0$$

so that the free surface AB corresponded to the arc of the circle $\zeta = e^{is}$ and the remaining part of the boundary has an effective diameter $\zeta = t$. In this way we locate the infinitely distant point C in accordance with the origin of the lamina coordinate ζ, while we determine point D by means of t_1 (Fig. 2).

The derived representation of the semicircle in the region of the complex potential $w(z) = \varphi(x, y) + i\psi(x, y)$

$$\frac{dw}{d\zeta} = Kq_0 \frac{(1-\zeta)(1+\zeta)^{\lambda}}{\zeta^{0.5(3+\lambda)}} \tag{1}$$

where K is a real constant, q_0 is the velocity at point A, and λ is equal to 0 and 1 for the systems of Ryabushinskii and Zhukovskii–Roshko, respectively.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhanik i Tekhnicheskoi Fiziki, No. 5, pp. 132-136, September-October, 1971. Original article submitted March 23, 1971.

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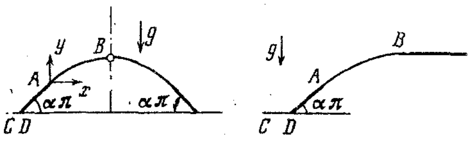

Fig. 1

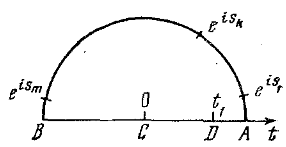

Fig. 2

We look for a complex velocity $dw/dz = q\exp(-i\theta)$ in the form

$$\frac{1}{q_0}\frac{dw}{dz} = e^{-i\alpha\pi}\left(\frac{\zeta - t_1}{1 - t_1\zeta}\right)^{\alpha} e^{M(\zeta)} \tag{2}$$

Then to determine the analytical function $M(\zeta)$ we arrive at the following boundary-value problem:

$$\operatorname{Re} M(e^{is}) = \ln\frac{q(s)}{q_0}, \quad s \in [0, \pi]; \ \operatorname{Im} M(t) = 0, \quad t \in [-1, 1]$$

Extending $M(\zeta)$ using the symmetry principle to the whole unit circle $|\zeta| \leq 1$ and taking into account the parity $q(s)$, we obtain, from the given function $M(\zeta)$ by the Schwartz equation,

$$M(\zeta) = \frac{1}{\pi}\int_0^{\pi} \ln\frac{q(s)}{q_0}\,\frac{1-\zeta^2}{1 - 2\zeta\cos s + \zeta^2}\,ds \tag{3}$$

The dependence $z = z(\zeta)$ is found from Eqs. (1) and (2)

$$z(\zeta) = Ke^{i\alpha\pi}\int_1^{\zeta} \frac{(1-\zeta)(1+\zeta)^{\lambda}}{\zeta^{0.5(3+\lambda)}}\left(\frac{1 - t_1\zeta}{\zeta - t_1}\right)^{\alpha} e^{-M(\zeta)}\,d\zeta \tag{4}$$

The pressure p at the boundary line of the flow is determined from the Bernoulli integral

$$p = p_* + \frac{\rho q_\infty^2}{2} - \frac{\rho q^2}{2}, \quad p_* = p_\infty - \rho g y$$

where p_∞ is the pressure in the undisturbed current at the level of point A, q_∞ is the current flow velocity, g is the accelerating force of gravity, and ρ is the fluid density. From here we seek in the stabilizing force of the pressure p_0 at the boundary AB a function q(s) which should satisfy the condition

$$\frac{q^2(s)}{q_0^2} = 1 - \frac{2}{(1+\sigma)\mathrm{Fr}^2}\,\frac{y(s)}{y_0} \quad \left(\sigma = \frac{p_\infty - p_0}{{}^1\!/_2\,\rho q_\infty^2} = \frac{q_0^2}{q_\infty^2} - 1, \ \mathrm{Fr} = \frac{q_\infty}{\sqrt{g y_0}}\right) \tag{5}$$

In this equation σ is the cavitation number and Fr is the Froude number determined from the quantity $y_0 = l\sin\alpha\pi$. We choose the dependence $q = q(s)$ so that (5) is satisfied at a finite number of points of the free boundary.

Let $\zeta_k = e^{is_k}$ ($k = 1, \ldots, m$) be images of some points $z_k = x_k + iy_k$ of the free boundary. In accordance with (5) the relative velocity q_k/q_0 at the points z_k is determined only through the physical parameters σ, Fr, and the dimension $l_k = y_k/y_0$

$$\frac{q_k}{q_0} = \left[1 - \frac{2l_k}{(1+\sigma)\mathrm{Fr}^2}\right]^{1/2} \quad (k = 1, \ldots, m+1) \tag{6}$$

The rates of flow q(s) at each of the intervals $[s_k, s_{k+1}]$ we assign according to the formula

$$\frac{q(s)}{q_0} = \left(\frac{q_k}{q_0}\right)^{\beta_{k+1}}\left(\frac{q_{k+1}}{q_0}\right)^{-\beta_k}, \quad \beta_{k+1} = \frac{\cos s - \cos s_{k+1}}{\cos s_k - \cos s_{k+1}}, \quad \beta_k = \frac{\cos s - \cos s_k}{\cos s_k - \cos s_{k+1}} \quad (s_k \leqslant s \leqslant s_{k+1}) \tag{7}$$

Then $q(s_k) = q_k$ and in the case where $q_{k+1} < q_k$ ($k = 0, \ldots, m$), q(s) is a monotonically decreasing function of the parameter $s \in [0, \pi]$.

Substituting Eq. (7) in (3) after evaluating the integrals entering into the given function $M(\zeta)$, we obtain

$$M(\zeta) = \frac{1}{\pi}\sum_{k=0}^{m}\frac{1}{\cos s_k - \cos s_{k+1}}\left\{\frac{1-\zeta^2}{2\zeta}(s_{k+1} - s_k)\ln\frac{q_{k+1}}{q_k} + i\left[\left(\frac{1+\zeta^2}{2\zeta} - \cos s_{k+1}\right)\ln\frac{q_k}{q_0} + \left(\cos s_k - \frac{1+\zeta^2}{2\zeta}\right)\ln\frac{q_{k+1}}{q_0}\right]\ln\frac{(\zeta - e^{is_k})(1 - e^{is_{k+1}}\zeta)}{(1 - e^{is_k}\zeta)(\zeta - e^{is_{k+1}})}\right\} \tag{8}$$

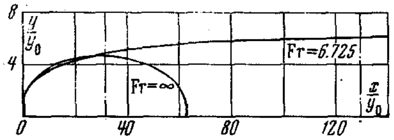

Fig. 3

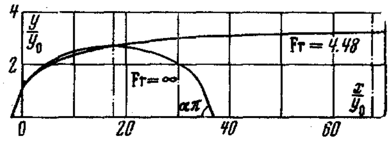

Fig. 4

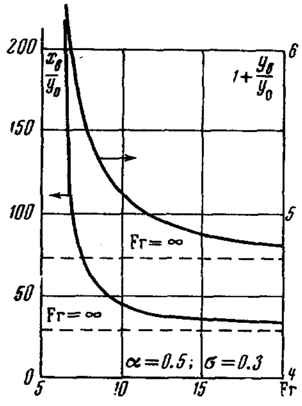

Fig. 5

The condition (5) at points $z_1, \ldots, z_m, z_{m+1}$ of the free boundary in accordance with (6) lead to the m + 1 equation

$$\frac{q_k^2}{q_0^2} = 1 - \frac{4}{\mathrm{Fr}^2 (1+\sigma) \sin \alpha\pi} \int_0^{s_k} q_0 \sin s/2 \, (2 \cos s/2)^{\lambda} \sin \theta(s) \frac{ds}{q(s)} \times \left[\int_{t_1}^{1} \frac{(1-t)(1+t)^{\lambda}}{t^{0.5(3+\lambda)}} \left(\frac{1-t_1 t}{t - t_1} \right)^{\alpha} e^{-M(t)} dt \right]^{-1} \tag{9}$$

From the condition for the velocity of an inflowing current

$$q_\infty = \left. \frac{dw}{dz} \right|_{\zeta=0}$$

it follows that one may express t_1 through the other parameters

$$t_1^{2\alpha} = \frac{1}{(1+\sigma) e^{2M(0)}}$$

Having now arbitrarily fixed the form of the parameter s_k (k = 1, . . . , m), we obtain from Eq. (9) the system of m + 1 equations for determination of the magnitude of q_k/q_0 (k = 1, . . . , m + 1).

In order to solve the system of equations (9) it is necessary that the given characteristics of flow σ and Fr satisfy the condition

$$\sigma \mathrm{Fr}^2 > 2y_b / y_0 \tag{10}$$

which arises during proof of the existence of a solution of the given system by V. N. Monakhov's method [1]. On the other hand, the inequality (10) is satisfied for those cavitation flows in which the static pressure at the free boundary is greater than the pressure p_0 in the cavity

$$p_* = p_\infty - \rho g y(s) > p_0$$

In accordance with Eq. (5) this implies that the velocity at the free flow lines is greater than the undisturbed current velocity.

The unknown parameters q_k/q_0 are determined by integration from (9). As a result of the specifics of the given problem two methods of successive approximations are proposed.

In accordance with the first method a solution is first found for the problem of satisfying the Bernoulli integral at two points of the free boundary A and B. This solution allows one to find those values of the parameters s_k (k = 1, . . . , m), which correspond to points of the free boundary obtained, with ordinates $y_k = k y_b^{(0)}/(m+1)$, where $y_b^{(0)}$ is a known ordinate of the point B. The null approximation for solution of the original system (9) is determined by the equation

$$\frac{q_k^{(0)}}{q_0} = \left[1 - \frac{2}{(1+\sigma)\mathrm{Fr}^2} \frac{k}{m+1} \frac{y_b^{(0)}}{y_0} \right]^{1/2} \quad (k = 1, \ldots, m+1)$$

and further approximations are sought by the usual iterational scheme. At each step of the iterational process the values of parameters s_k chosen earlier are written unchanged. In the indicated method the process stops when the Froude number is close to the minimum $\mathrm{Fr}_* = 2y_b/\sigma y_0$.

In this interval of Froude numbers the second method is applied, in accordance with which additional iterations are carried out with respect to Fr. The solution of system (9) obtained is taken as the initial approximation to the solution of the given system for smaller Froude numbers, etc.

If parameters q_k/q_0 (k = 1, . . . , m + 1) are found satisfying the system of equations (9), then the flow is determined by Eqs. (1), (2), (4), and (8). The condition of stability of pressure in this case will be satisfied at m + 2 points of the current line AB. At the remaining points of this boundary the departure of the observed pressure distribution from the stable pressure p_0 is characterized by the magnitude

$$\tau(s) = 1 - \frac{p - p_\infty}{p_0 - p_\infty} = \left(1 + \frac{1}{\sigma}\right)\left[1 - \frac{q^2(s)}{q_0^{\,2}}\right] - \frac{2}{\sigma\,\mathrm{Fr}^2}\,\frac{y(s)}{y_0}$$

Determining for each of these intervals $[s_k, s_{k+1}]$, $k = 0, \ldots, m$ the points of the extremum $\tau(s)$, one may then find a minimum value τ_* of the function $\tau(s)$ for all the intervals $0 \le s \le \pi$. The required smallness of the quantity τ_* is achieved at the expense of an increase in the number of points of the free boundary at which the boundary condition (5) is satisfied.

The results of calculations of the cavitation streamline flow of a lamina by Ryabushinskii's system for $\sigma = 0.3$ are presented in Figs. 3-5. In the solution of the problem the condition of pressure uniformity (5) for $m = 19$ is satisfied in all cases with an error τ_* not exceeding 0.1%.

A comparison of the form of the free boundary for the flow of a weightless and of a heavy fluid with small Froude numbers and at equal cavitation numbers for two values of the angle of inclination of the lamina $\alpha = 1/2, 1/6$ is presented in Figs. 3 and 4, respectively. The broken lines in these graphs mark the midsection of the cavity. The fact should be kept in mind that in the central part of the cavity the free flow line is somewhat higher for a weightless than for a heavy fluid.

The nature of the dependence of cavity size on the Froude number for a lamina normal for an inflowing current is illustrated in the graph of Fig. 5. To a considerable degree the effect of weight (floating of the cavity and an increase in its length) appears at small Froude numbers. It is seen from the graph that as Fr approaches $\mathrm{Fr}_* \approx 6.15$ the length of the cavity increases considerably, while its width (maximum transverse size) is limited below some fixed size.

Analogous calculations were made for the case of cavitation streamline flow of a lamina at the lower half-surface. Under the same conditions the length and width of the cavity is greater at the lower half-surface than at the upper.

Of great interest is the case where the net lifting force produced on the body of the cavity on the part of the liquid is equal to the weight of its displaced fluid. The method of formation of this type of flow was worked out by B. G. Novikov.

Suppose a solution is found for the problem of cavitation streamline flow of a lamina by Ryabushinskii's system at the upper (lower) half-surface. The condition $\psi = 0$ at the boundary line of the flow allows one to extend the function w using the symmetry principle through the horizontal base $y = 0$ into the lower (upper) half-surface. Then the flow, at all surface under investigation, will be symmetrical and, consequently, noncirculating. Since at the points $P(x, y)$ and $P(x, -y)$, lying on the free boundary and on its mirror image relative to the direction $y = 0$, the velocities are equal, it follows from the Bernoulli integral that the pressures at the symmetrical points differ by the amount $2\rho gy$. Hence the resultant lifting force produced on the part of the fluid on the partially cavitated body will be equal to the ejecting Archimedes force.

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